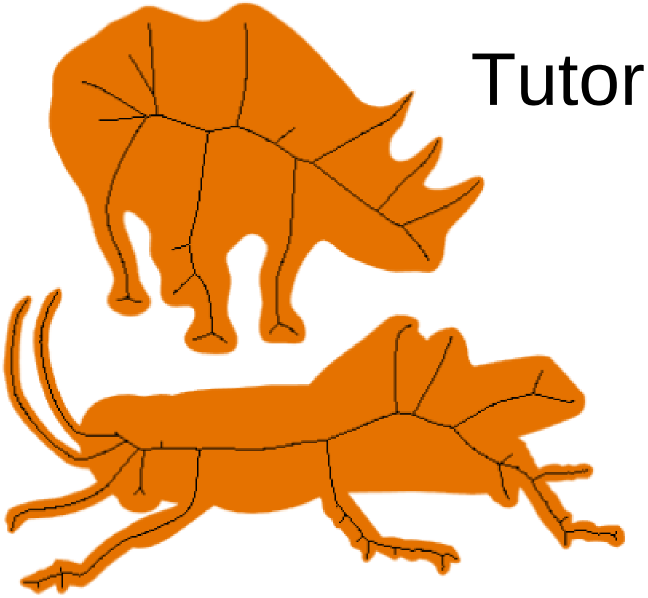
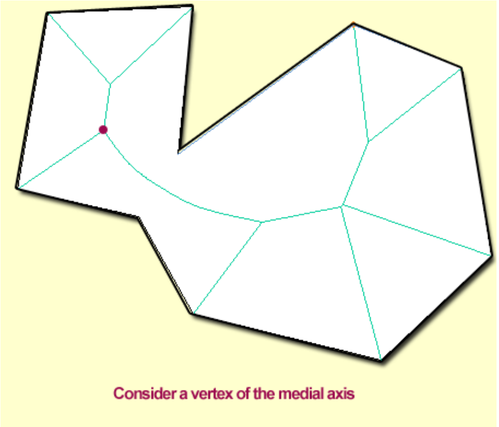
Polygonal Skeletons

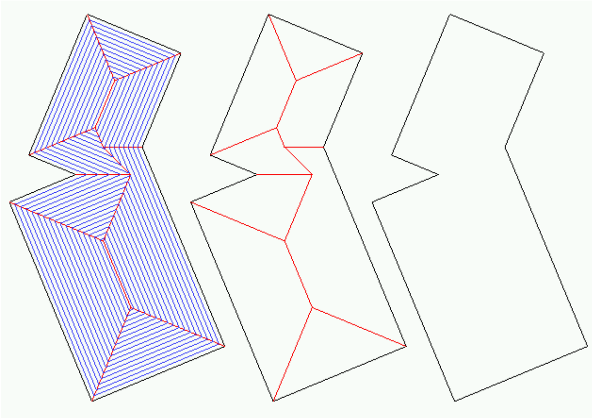
Tutorial 2 – Computational Geometry



The Skeleton of a Simple Polygon

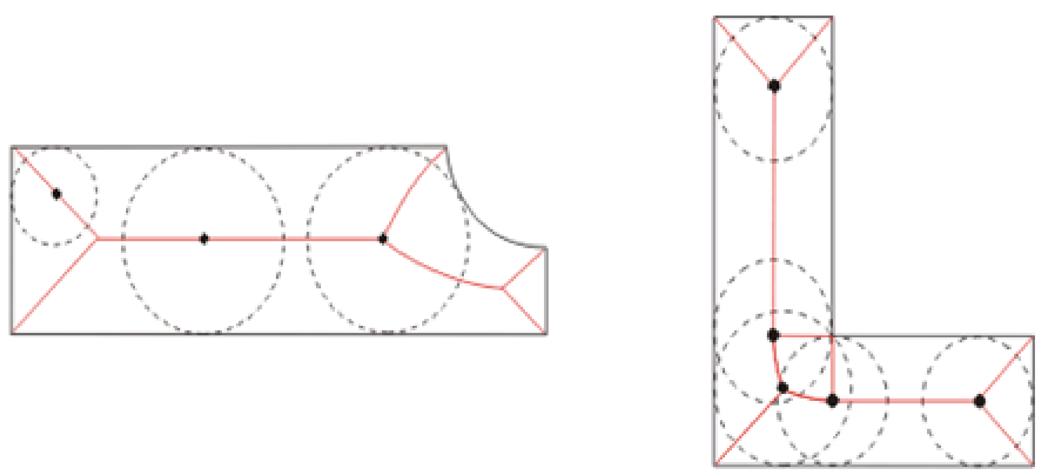
- A **polygon** is a closed contour in the plane, which might contain holes (which are simple polygons as well).
- A ***skeleton*** of a polygon is a partition of the polygon into regions, creating internal vertices, edges and faces.
- We will deal with two main types of skeletons: **The Medial Axis and the Straight skeleton.**



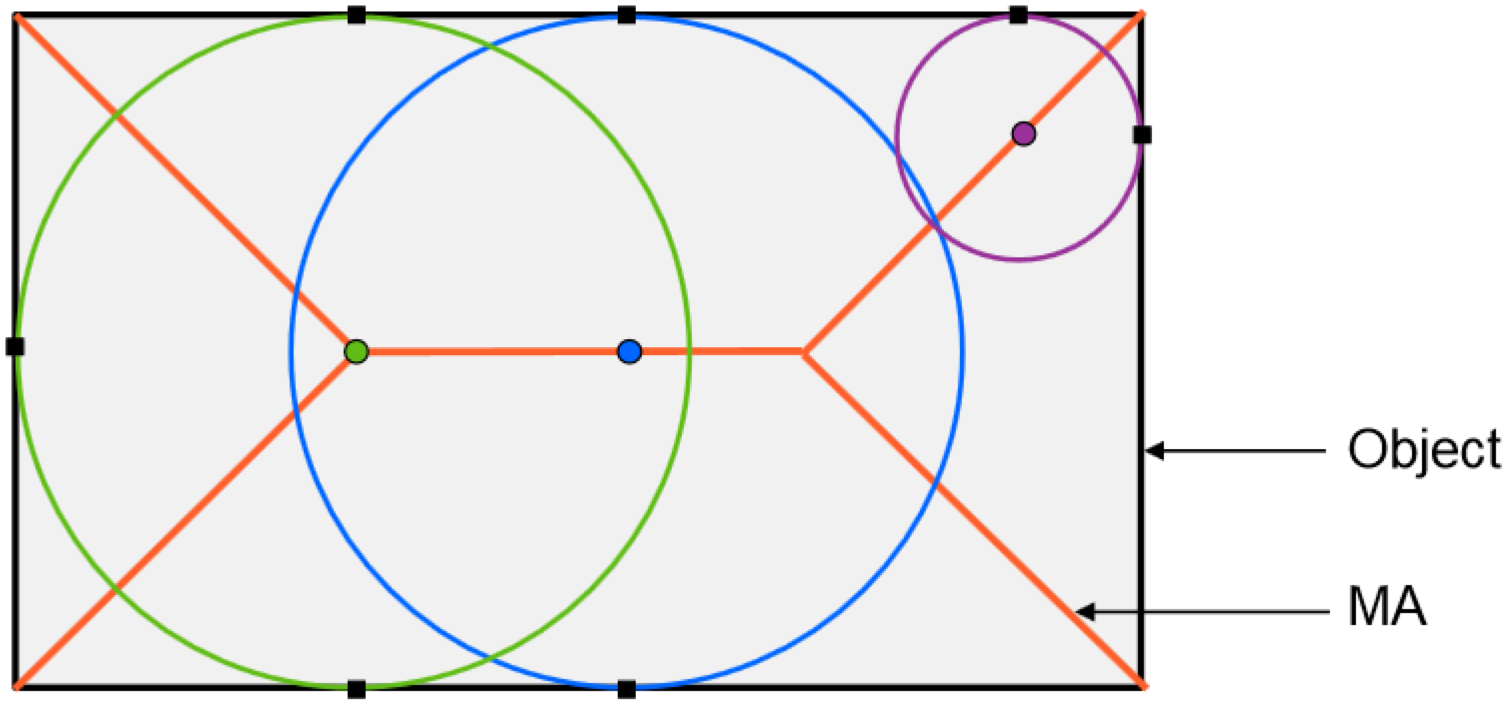


The Medial Axis

- **<u>The Medial axis:</u>** the locus of the centers of circles that are tangent to the polygon at two or more points.
- **<u>locus:</u>** a set of points whose location satisfies one or more specified conditions.

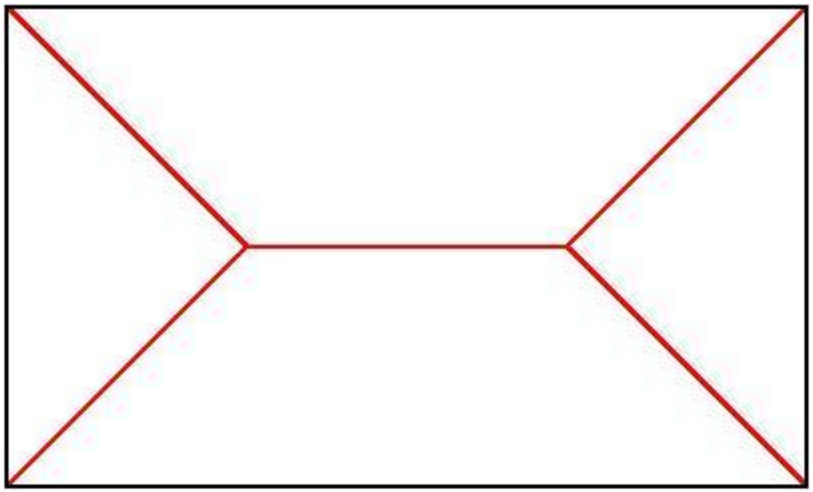


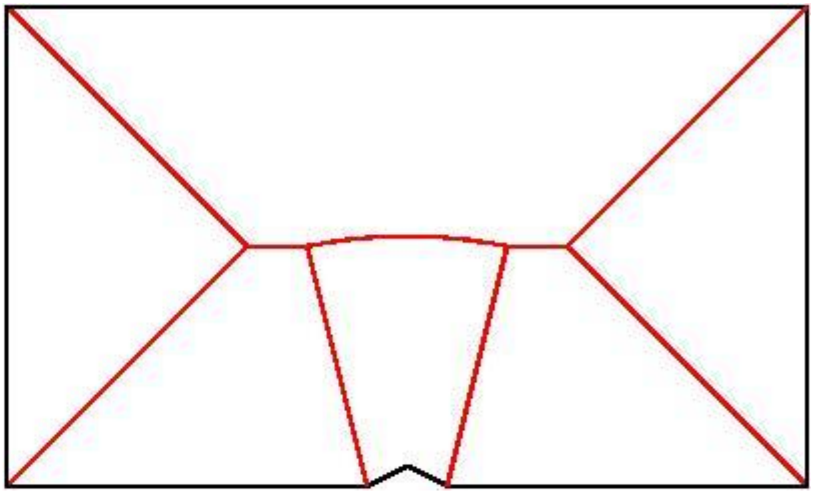
The Medial Axis: Example



The Medial Axis – Continued

- The Medial Axis comprises straight lines if the polygon is convex.
- If the object is concave, every reflex vertex induces a curved edge.

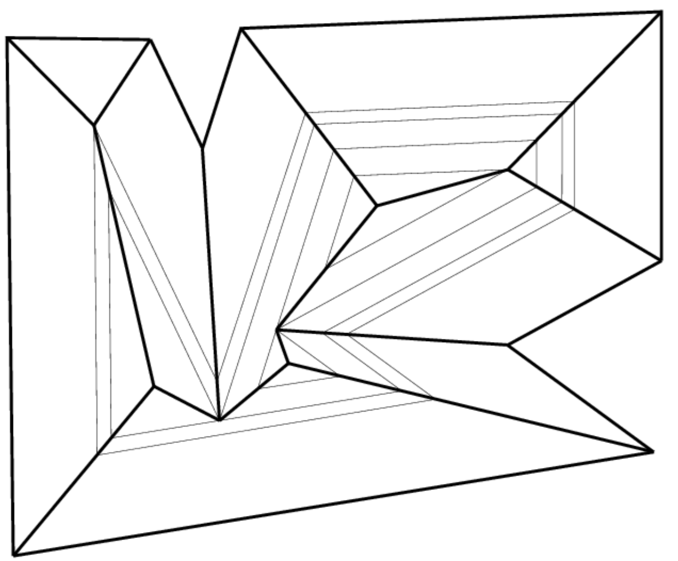




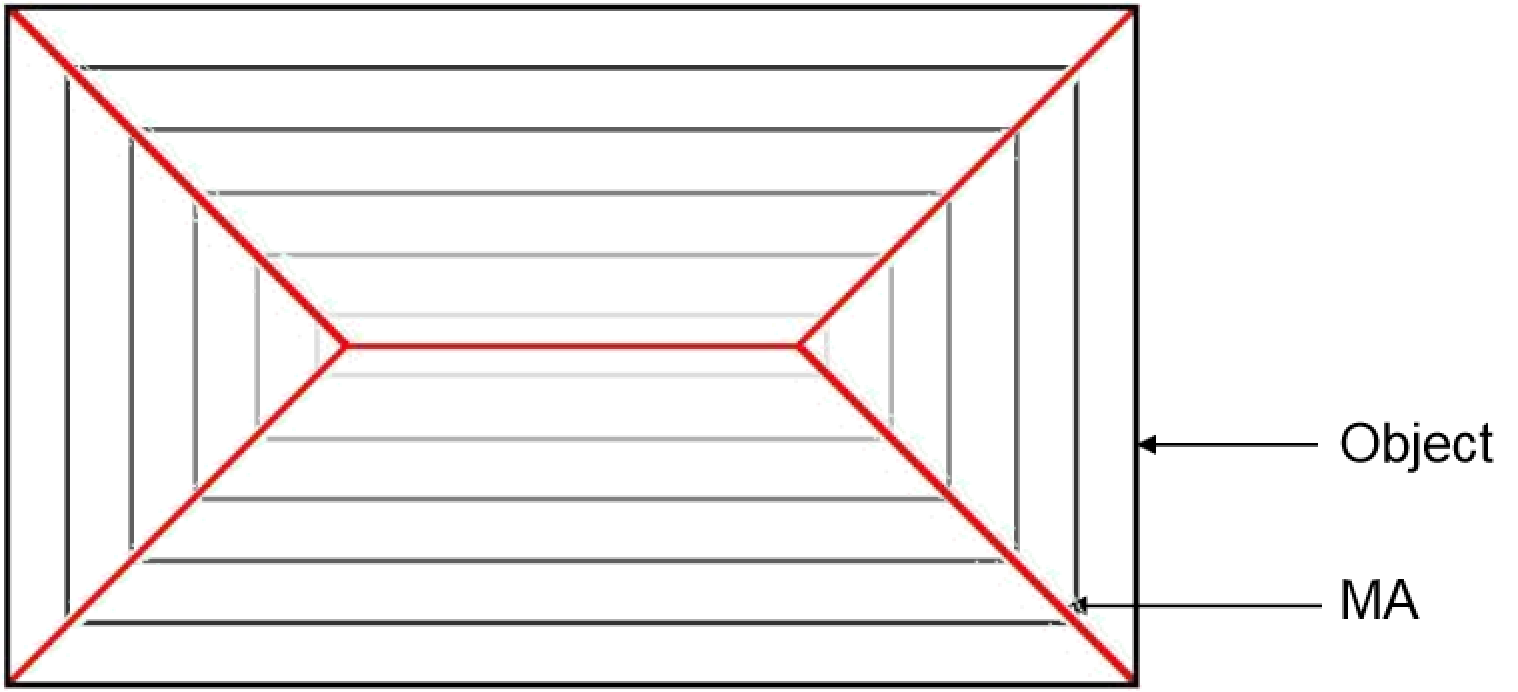
Another Problem

The Straight Skeleton

- The Straight Skeleton: the **trace** of the *angular bisectors* of the vertices, as the edges of the polygon are propagating at equal rate, until the polygon vanishes.
- It is a **linear approximation** of the Medial Axis.



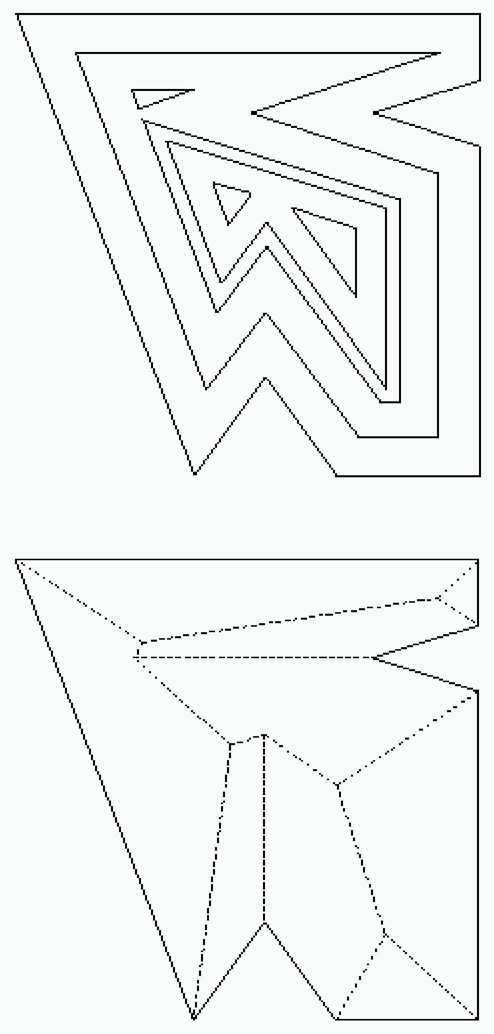
The Straight Skeleton



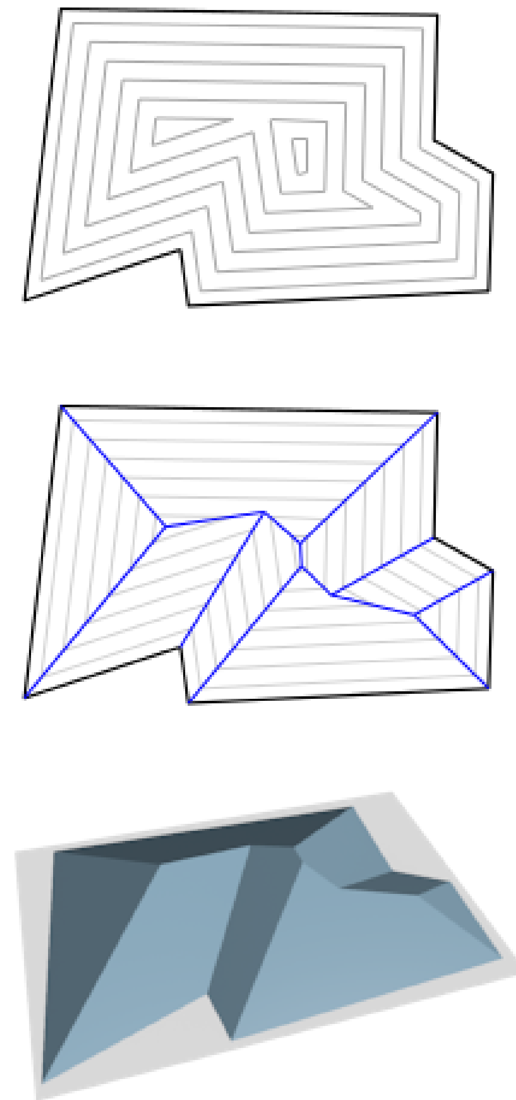
The Propagation of The Polygon

- As the edges of the polygon propagate at equal rate, the vertices move along the bisector of its two adjacent edges.

- **Two possible events** (assuming g.p.) may occur during the propagation:
  - **Edge Event**– A portion (or the whole) of an edge vanishes.
  - **Split Event**– A reflex vertex hits an opposite edge, splitting the polygon into two disconnected parts.



An Application of The Straight Skeleton

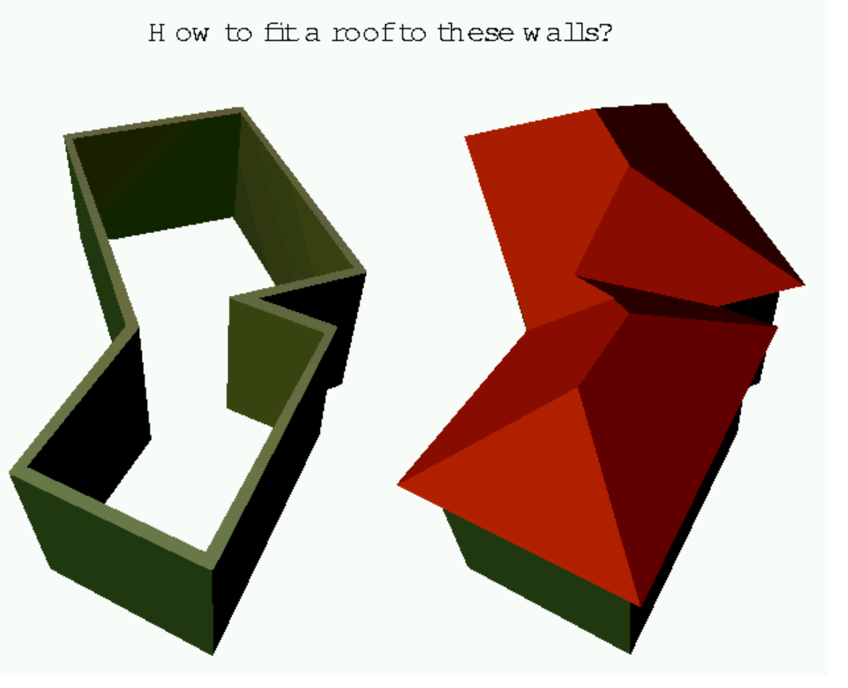


The Properties of The Straight Skeleton

- The faces of the straight skeleton are monotone (why?).
- Every internal skeleton node has degree 3*
- The Medial Axis and the straight skeleton of a convex polygon are identical.
- There are $2n-3$ edges, $n-2$ inner vertices and n faces in a straight skeleton.

Designing Rooftops

- When assigning a height field to an inner node - its offset distance from the edge - the skeleton can be interpreted as the rooftop of a house which walls are the sides of the original polygon.



Straight-Skeleton Computation

- Most algorithms take a straight-forward approach of event-based simulation of the propagation.
- The most time-efficient algorithm known has time complexity $O(n^{1+\varepsilon} + n^{8/11+\varepsilon} r^{9/11+\varepsilon})$

  r =# reflex vertices

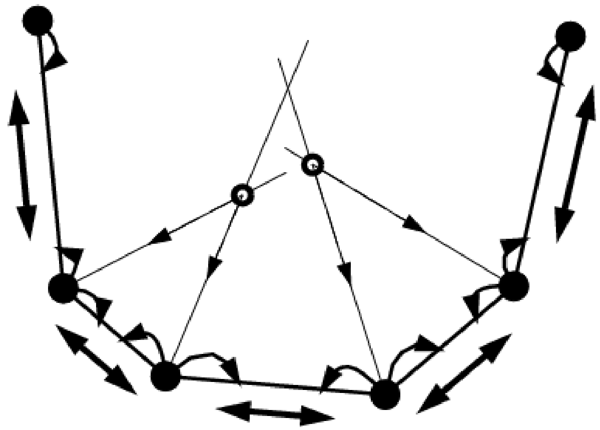
  n =# vertices

Felkel & Obdrzálek 98'

- Felkel & Obdrzálek offered a straightforward **event-based** algorithm.
- The algorithm computes and simulates the events by maintaining a set of circular **Lists of Active Vertices** called **LAV**s.
- The algorithm does not construct the intermediate offset polygons (although easily deduced), but only the skeleton itself.

The algorithm for Convex Polygons

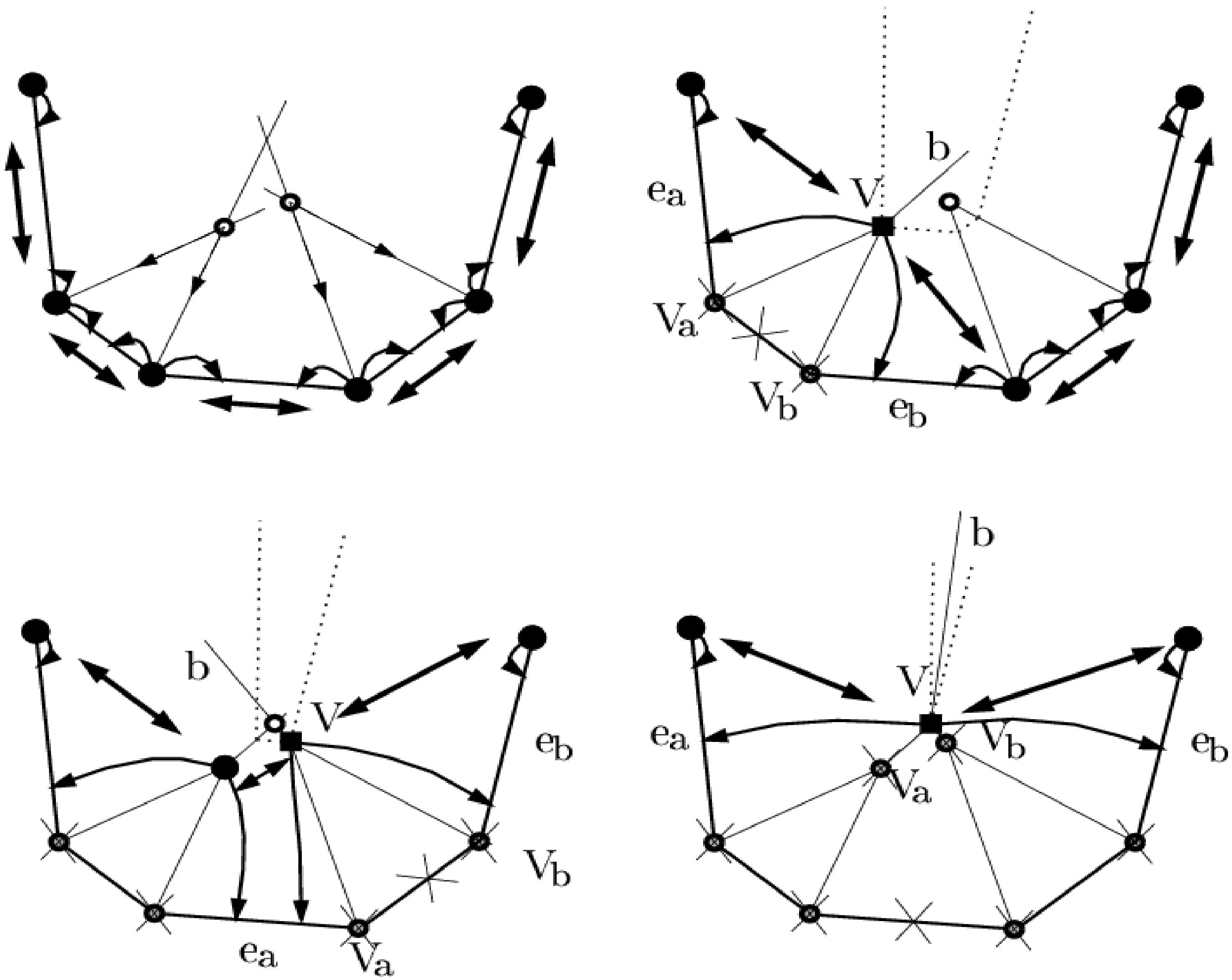
- **Initialization**
  - Create a **LAV** for the polygon – a circular list of its vertices by order.
  - Add pointers for the edges between vertices.
  - Compute a bisector per vertex.
  - All vertices are marked "*unused*".
- **Calculation of initial edge events**
  - Compute the intersection point of every set of adjacent bisectors – this point is the location of the edge event between them.
  - Queue the edge event (marked EDGE_EVENT) in a priority queue according to the distance of the intersection from the line supporting the edge.



Propagation Step

- While the events queue != empty do
  - If next event uses used vertices, discard event.
  - Else, handle edge event
    - If LAV contains more than 3 edges
      - Create two edges of the skeleton, each from one of the event vertices to the location of the event (the intersection point).
      - Remove these two vertices from the LAV, and mark them as “used”.
      - Create a new vertex, located at the intersection point, and put it in its place in the LAV, pointing to its adjacent edges.
      - Compute new edge events for the vertices of these adjacent edges.
    - Else, create new vertex at the intersection, and skeletal edges from each of the 3 vertices.

Propagation



Complexity

- The number of vertices reduces to zero, and the algorithm always stops.
- Complexity: $O(nlogn)$, for maintaining the events queue. Every event handling is $O(1)$.

The Algorithm for Nonconvex Polygons

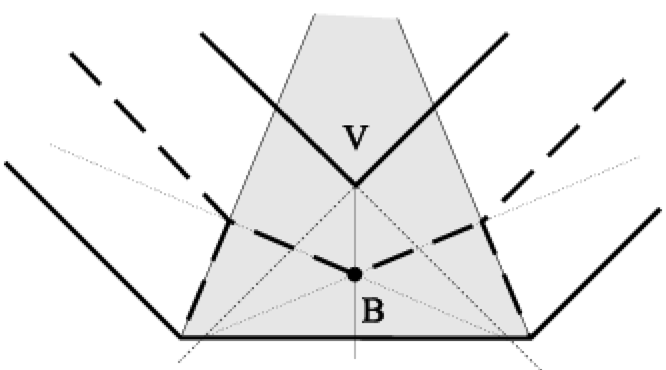
- An extension of the convex algorithm.
- We have to find out when split events occur.
- Another step in initialization:
  - Determine all possibilities of a reflex vertex hitting an opposite edge.
  - Queue these events as SPLIT_EVENT

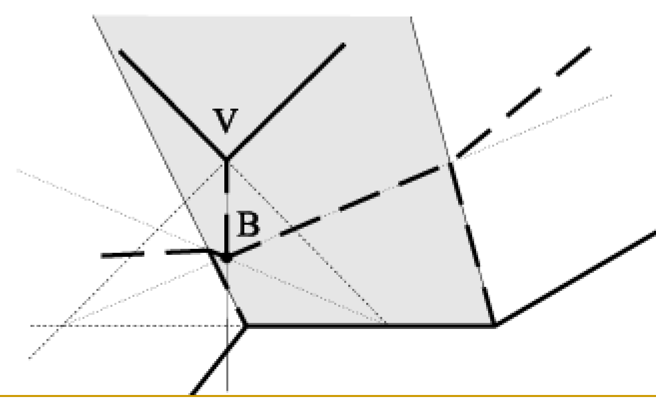
Obtaining Split Events

- A splitting location B is equidistant from:
  - the lines supporting the edges adjacent to the reflex vertex, and;
  - the line supporting the opposite edge.
- For every reflex vertex, we traverse all of the edges in the polygon and test for intersection.
  - A simple intersection test between the bisector of the reflex vertex and the opposite edge isn’t enough (why?).

Obtaining Split Events – Continued

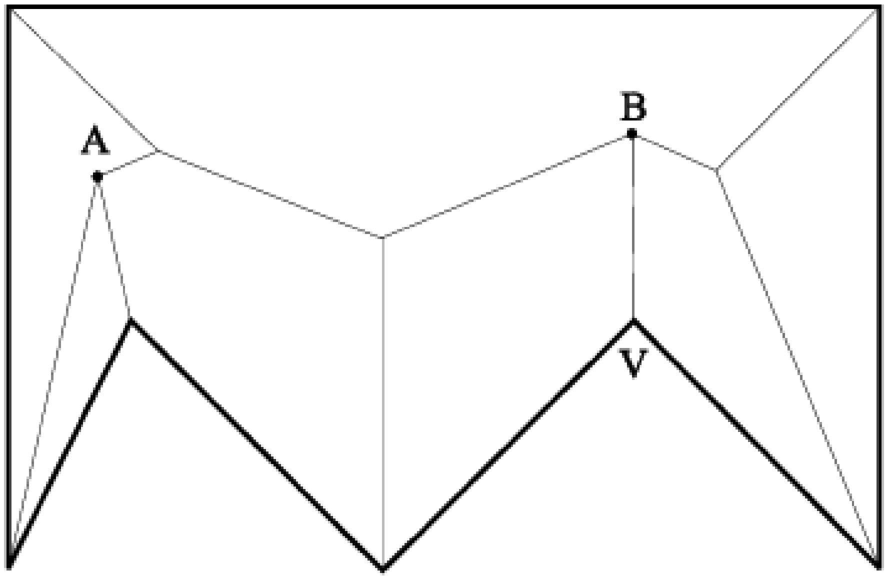
- The intersection point between the reflex vertex and the line supporting the opposite edges must be in the area defined between the edge and the bisectors of its two vertices.
- The intersection point is the meeting point of the three bisectors between all three participating edges (the two defining the reflex vertex and the split edge).





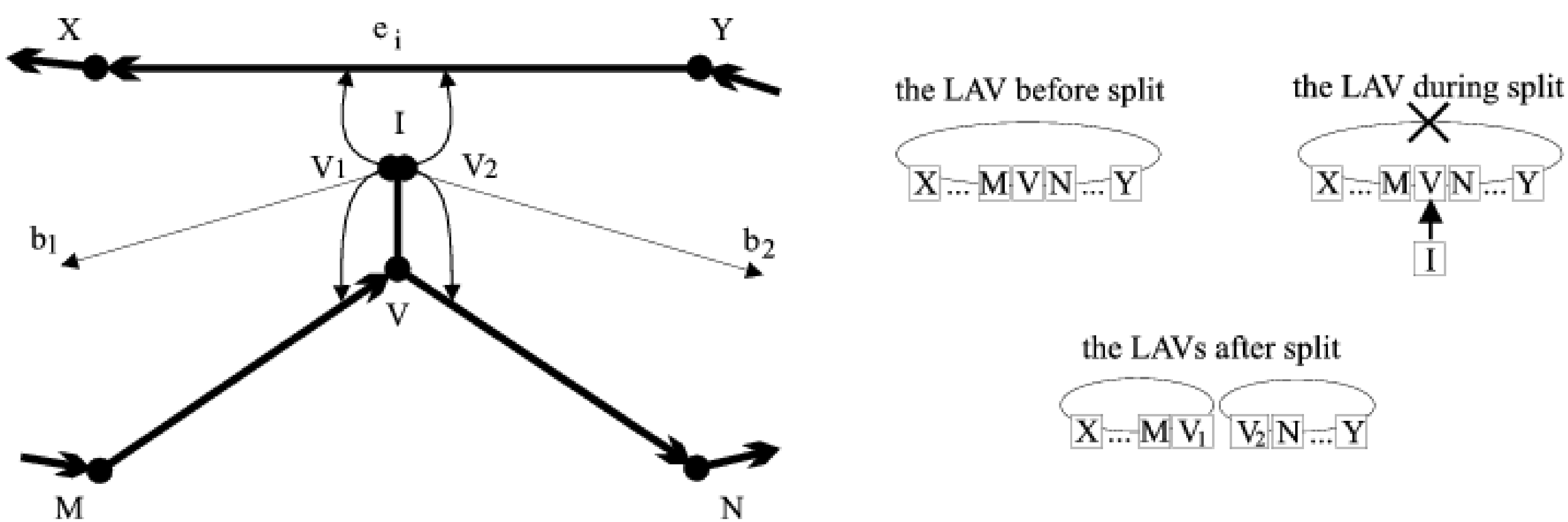
Obtaining Split Events

- Not all reflex vertices eventually cause split events. (A is an edge event, and B is a split event).



Handling Split Events

- When a split event occurs, the polygon splits into two parts.
- The LAV in context is split into two LAVs as well.

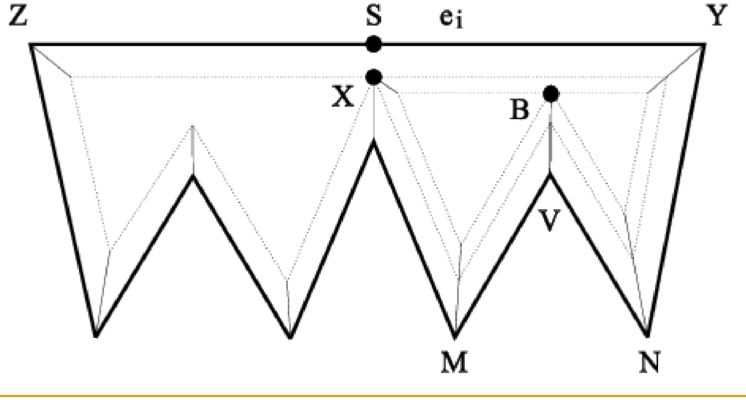


Handling Split Events – Cont'd

- The splitting vertex is replaced with two new vertices, each in the appropriate place in a different LAV.
- New bisectors and edge events are calculated for each of these vertices (why only edge events?)
- The propagation continues…

Handling Multiple Splitting

- An edge can be split several time.
- Any split event handling must realize what part of the edge it is splitting (i.e. what are the proper endpoints).
- It is done by traversing the LAV in context at each handling of a split event.



Summary of the General Algorithm

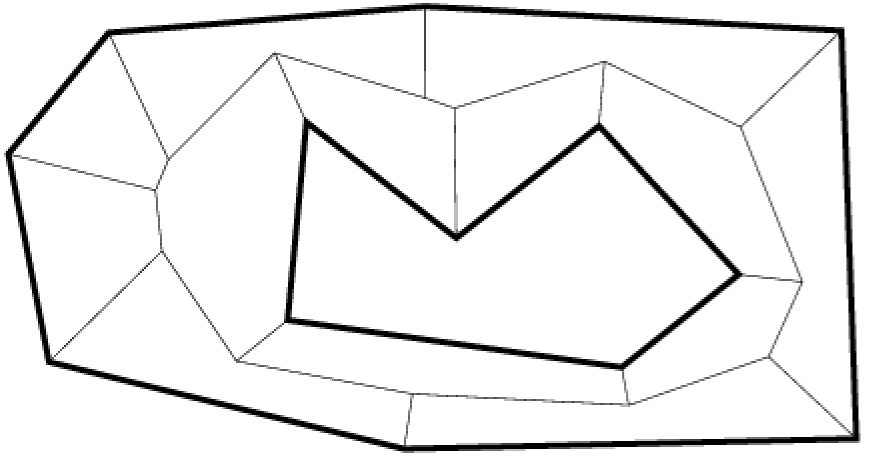
- **<u>Initialization</u>**
  - Create one LAV
  - Compute bisectors
  - Compute split and edge events
  - Queue all events according to time (distance)

Summary – Continued

- **<u>Propagation</u>**
  - While event queue has events
    - If new event contains used vertices, discard event.
    - If event is edge event, handle as in the convex case. Mark vertices as “used”. If the LAV in context contains 3 vertices, close up the skeleton.
    - If event is split event, split the LAV into two, and maintain pointers accordingly. Mark the splitting vertex as “used”.
- In the end, there are no LAVs left!

A Simple Polygon with Holes

- The approach is similar.
- Any hole is a different LAV in the initialization.
- Two LAVs can merge when a split event occurs between two different boundaries – correct LAV pointer treatment should be applied.

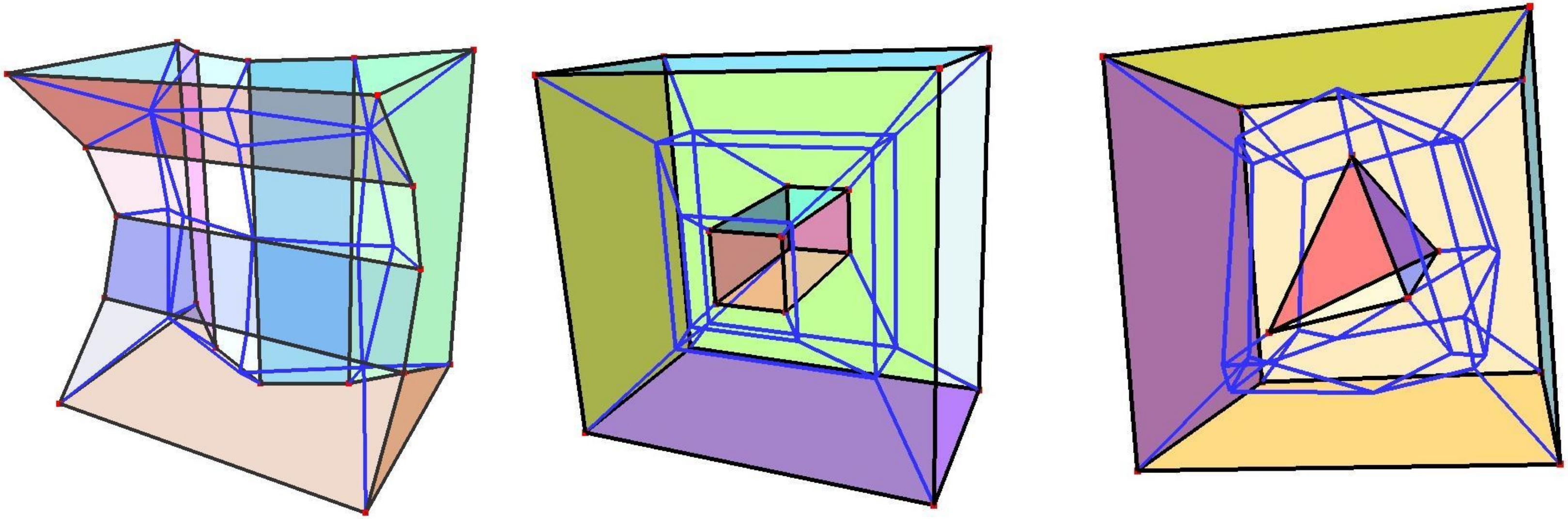


The Complexity of the Algorithm

- Initializing and handling each split event require traversing all of the edges per each reflex vertex.
- So, the total complexity is $O(rn + nlogn)$.
  - r = # reflex vertices
- If $r = O(n)$ then the algorithm is quadratic – a reachable upper bound.
- Most practical cases behave better.
- Space complexity is *O(n)*.

3D Straight skeletons – A (New) View

- The faces of a polyhedron propagate at equal rate.
- Skeleton is the trace of faces, edges and vertices.



Bibliography

- Source of images (and recommended reading):
  - "Medial Axis presentation" - http://groups.csail.mit.edu/graphics/classes/6.838/F01/lectures/MedialAxisEtc/presentation/
  - "Single-Fold Disk Hiding" - http://jeff.cs.mcgill.ca/~mcleish/507/single.html
  - "Straight skeleton of a simple polygon" - http://compgeom.cs.uiuc.edu/~jeffe/open/skeleton.html
  - "Raising roofs, crashing cycles, and playing pool" - http://compgeom.cs.uiuc.edu/~jeffe/pubs/cycles.html
  - "Designing Roofs of Buildings " - http://www.sable.mcgill.ca/~dbelan2/roofs/roofs.html
- Straight Skeleton Computation
  - P. Felkel and S. Obdrzalek, Straight skeleton computation, Spring Conf. on Computer Graphics, Budmerice, Slovakia, 210--218, 1998.